

# The Atiyah-Patodi-Singer index and domain-wall fermion Dirac operators

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This talk is based on a joint work

[arXiv:1910.01987](https://arxiv.org/abs/1910.01987) (to appear in CMP)

of three mathematicians and three physicists:

- Mikio Furuta
- Mayuko Yamashita
- Shinichiroh Matsuo
- Hidenori Fukaya
- Tetsuya Onogi
- Satoshi Yamaguchi

# Main theorem

## Theorem (FFMOYY arXiv:1910.01987)

For  $m \gg 0$ , we have a formula

$$\text{Ind}_{\text{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

- The Atiyah-Patodi-Singer index is expressed in terms of the  $\eta$ -invariant of **domain-wall fermion Dirac operators**.
- The original motivation comes from the bulk-edge correspondence of topological insulators in condensed matter physics.
- The proof is based on a Witten localisation argument.

# Plan of the talk

1. Reviews of the Atiyah-Singer index and the eta invariant
2. The Atiyah-Patodi-Singer index
3. Domain-wall fermion Dirac operators
4. Main theorem
5. The proof of a toy model
6. The proof of the main theorem: Witten localisation

## Index and Eta

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Let  $X$  be a closed manifold and  $S \rightarrow X$  a hermitian bundle. Assume  $\dim X$  is even. Assume  $S$  is  $\mathbb{Z}/2$ -graded: there exists  $\gamma: \Gamma(S) \rightarrow \Gamma(S)$  such that  $\gamma^2 = \text{id}_S$ .

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let  $D: \Gamma(S) \rightarrow \Gamma(S)$  be a 1st order elliptic differential operator. Assume  $D$  is **odd** and **self-adjoint**:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \text{ and } D_- = (D_+)^*.$$

### Definition

$$\begin{aligned} \text{Ind } D &:= \dim \text{Ker } D_+ - \dim \text{Ker } D_- \\ &= \dim \text{Ker } D_+ - \dim \text{Coker } D_+ \end{aligned}$$

Fix  $m \neq 0$  and consider

$$D + m\gamma = \begin{pmatrix} m & D_- \\ D_+ & -m \end{pmatrix} : \Gamma(S) \rightarrow \Gamma(S).$$

This is self-adjoint but no longer odd; thus, its spectrum is real but not symmetric around 0. For  $\operatorname{Re}(z) \gg 0$ , let

$$\eta(D + m\gamma)(z) := \sum_{\lambda_j} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z},$$

where  $\{\lambda_j\} = \operatorname{Spec}(D + m\gamma)$ . Note that  $\lambda_j \neq 0$  for any  $j$ .

### Definition

$$\eta(D + m\gamma) := \eta(D + m\gamma)(0).$$

The eta invariant describes the overall asymmetry of the spectrum of a self-adjoint operator.

## Proposition

For any  $m > 0$ , we have a formula

$$\text{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

This formula might be unfamiliar; however, we can prove it easily, for example, by diagonalising  $D^2$  and  $\gamma$  simultaneously. We will explain another proof later.

We will generalise this formula to handle compact manifolds with boundary and **the Atiyah-Patodi-Singer index**.



## Proposition

For any  $m > 0$ , we have a formula

$$\text{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

We will generalise this formula to handle compact manifolds with boundary and the Atiyah-Patodi-Singer index by using **domain-wall fermion Dirac operators**.

## Theorem (FFMOYY arXiv:1910.01987)

For  $m \gg 0$ , we have a formula

$$\text{Ind}_{\text{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

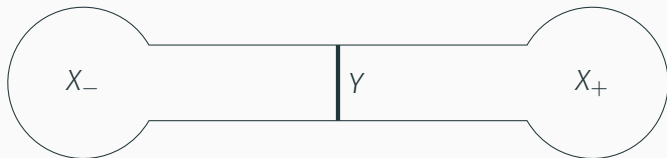
Next, we review the Atiyah-Patodi-Singer index.

# The Atiyah-Patodi-Singer index

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Let  $Y \subset X$  be a separating submanifold that decomposes  $X$  into two compact manifolds  $X_+$  and  $X_-$  with common boundary  $Y$ . Assume  $Y$  has a collar neighbourhood isometric to  $(-4, 4) \times Y$ .

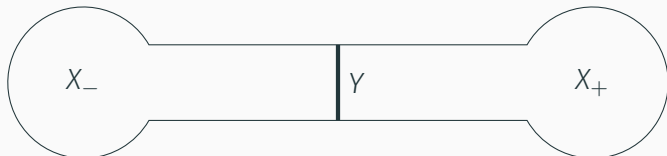
$$(-4, 4) \times Y \subset X = X_- \cup_Y X_+$$



Assume  $S \rightarrow X$  and  $D: \Gamma(S) \rightarrow \Gamma(S)$  are standard on  $(-4, 4) \times Y$  in the sense that there exists a hermitian bundle  $E \rightarrow Y$  and a self-adjoint elliptic operator  $A: \Gamma(E) \rightarrow \Gamma(E)$  such that  $S = \mathbb{C}^2 \otimes E$  and

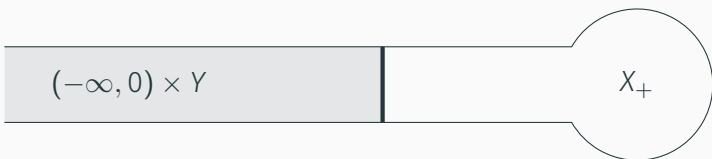
$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix}$$

on  $(-4, 4) \times Y$ .



Assume also  $A$  has no zero eigenvalues.

Let  $\widehat{X}_+ := (-\infty, 0] \times Y \cup X_+$ .



We assumed  $D$  is translation invariant on  $(-4, 4) \times Y$ :

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix}.$$

Thus,  $D|_{X_+}$  naturally extends to  $\widehat{X}_+$ , which is denoted by  $\widehat{D}$ .

This is Fredholm if  $A$  has no zero eigenvalues.

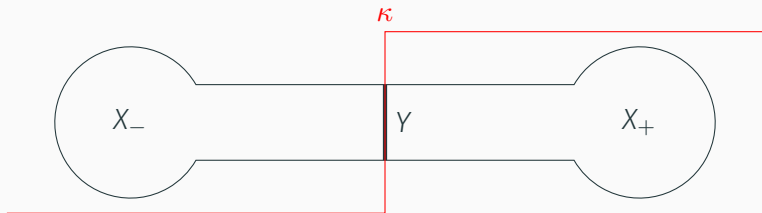
**Definition (Atiyah-Patodi-Singer index)**

$$\text{Ind}_{\text{APS}}(D|_{X_+}) := \text{Ind}(\widehat{D})$$

# Domain-wall fermion Dirac operators

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Let  $\kappa: X \rightarrow \mathbb{R}$  be a step function such that  $\kappa \equiv \pm 1$  on  $X_{\pm}$ .



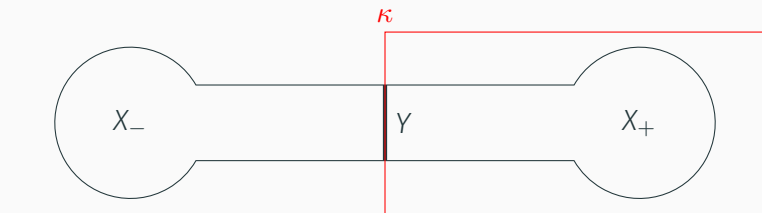
### Definition

For  $m > 0$ ,

$$D + m\kappa\gamma: \Gamma(S) \rightarrow \Gamma(S)$$

is called a **domain-wall fermion Dirac operator**.

$D + m\kappa\gamma$  is self-adjoint but not odd.



$$D = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix} \text{ on } (-4, 4) \times Y$$

### Proposition

If  $\text{Ker} A = \{0\}$ , then  $\text{Ker}(D + m\kappa\gamma) = \{0\}$  for  $m \gg 0$ .

Next we will define  $\eta(D + m\kappa\gamma)$ .



# The eta invariant of domain-wall fermion Dirac operators

Since  $\text{Ker}(D + m\kappa\gamma) = \{0\}$ , there exists a constant  $C_m > 0$  such that  $\text{Ker}(D + m\kappa\gamma + f) = \{0\}$  if  $\|f\|_2 < C_m$ .

## Corollary of the variational formula of the eta invariant

Assume both  $m\kappa\gamma + f_1$  and  $m\kappa\gamma + f_2$  are smooth with  $\|f_1\|_2 < C_m$  and  $\|f_2\|_2 < C_m$ . Then, we have

$$\eta(D + m\kappa\gamma + f_1) = \eta(D + m\kappa\gamma + f_2).$$

## Definition

For any  $f$  with  $\|f\|_2 < C_m$  and  $m\kappa\gamma + f$  smooth, we set

$$\eta(D + m\kappa\gamma) := \eta(D + m\kappa\gamma + f).$$

## Main theorem

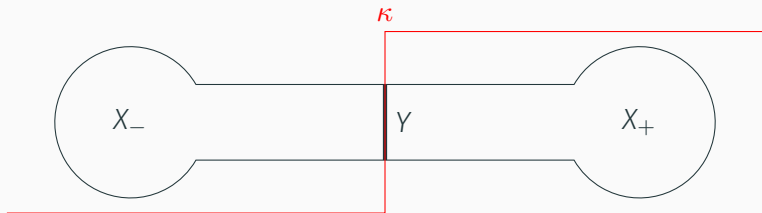
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# Main theorem

Theorem (FFMOYY arXiv:1910.01987)

For  $m \gg 0$ , we have a formula

$$\text{Ind}_{\text{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$



- The Atiyah-Patodi-Singer index is expressed in terms of the  $\eta$ -invariant of domain-wall fermion Dirac operators.
- The original motivation comes from physics.
- The proof is based on a Witten localisation argument.

## The proof of a toy model

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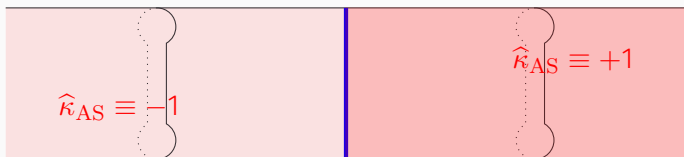
## Proposition

*For any  $m > 0$ , we have a formula*

$$\text{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

As a warm-up, we will prove this formula in the spirit of our proof of the main theorem.

Let  $\widehat{\kappa}_{AS}: \mathbb{R} \times X \rightarrow \mathbb{R}$  be a step function such that  $\widehat{\kappa}_{AS} \equiv 1$  on  $(0, \infty) \times X$  and  $\widehat{\kappa}_{AS} \equiv -1$  on  $(-\infty, 0) \times X$ .



We consider  $\widehat{D}_m: L^2(\mathbb{R} \times X; S \oplus S) \rightarrow L^2(\mathbb{R} \times X; S \oplus S)$  defined by

$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{AS}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{AS}\gamma) - \partial_t & 0 \end{pmatrix}.$$

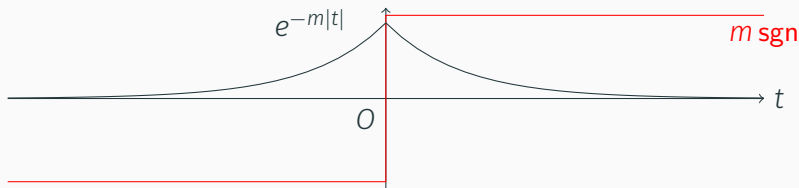
This is a Fredholm operator.

## Model case: the Jackiw-Rebbi solution on $\mathbb{R}$

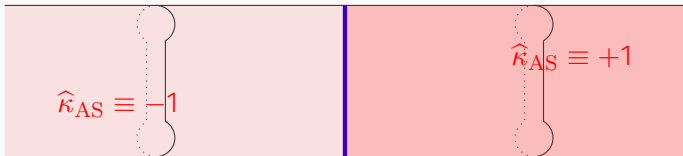
For any  $m > 0$ , we have

$$\frac{d}{dt} e^{-m|t|} = -m \operatorname{sgn} e^{-m|t|},$$

where  $\operatorname{sgn}(\pm t) = \pm 1$ . As  $m \rightarrow \infty$ , the solution concentrates at 0.



$$\begin{pmatrix} 0 & \partial_t + m \operatorname{sgn} \\ -\partial_t + m \operatorname{sgn} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e^{-m|t|} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$



$$\hat{D}_m := \begin{pmatrix} 0 & (D + m\hat{\kappa}_{AS}\gamma) + \partial_t \\ (D + m\hat{\kappa}_{AS}\gamma) - \partial_t & 0 \end{pmatrix}$$

$$(e^{-m|t|})' = -m \operatorname{sgn} e^{-m|t|}$$

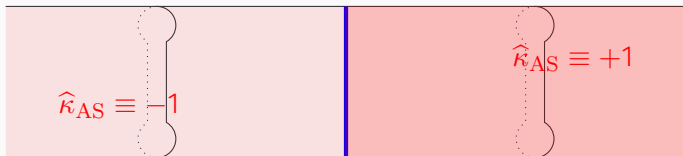
**Proposition (Product formula)**

$$\operatorname{Ind}(D) = \operatorname{Ind}(\hat{D}_m)$$

Assume  $D\phi = 0$ . Set  $\phi_{\pm} := (\phi \pm \gamma\phi)/2$ . Then, we have

$$\begin{pmatrix} 0 & (D + m\hat{\kappa}_{AS}\gamma) + \partial_t \\ (D + m\hat{\kappa}_{AS}\gamma) - \partial_t & 0 \end{pmatrix} \begin{pmatrix} e^{-m|t|}\phi_{-} \\ e^{-m|t|}\phi_{+} \end{pmatrix} = 0.$$





$$\hat{D}_m := \begin{pmatrix} 0 & (D + m\hat{\kappa}_{AS}\gamma) + \partial_t \\ (D + m\hat{\kappa}_{AS}\gamma) - \partial_t & 0 \end{pmatrix}$$

### Proposition (APS formula)

$$\text{Ind}(\hat{D}_m) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}$$

- Note that  $D + m\hat{\kappa}_{AS}(\pm 1, \cdot)\gamma = D \pm m\gamma$ .
- Perturb  $\hat{\kappa}_{AS}$  slightly near  $\{0\} \times X$  to get a smooth operator.
- Use the Atiyah-Patodi-Singer index theorem on  $\mathbb{R} \times X$ .
- Since  $\dim \mathbb{R} \times X$  is odd, the constant term in the asymptotic expansion of the heat kernel vanishes.

## Proposition

$$\text{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{AS}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{AS}\gamma) - \partial_t & 0 \end{pmatrix}$$

By the product formula, we have

$$\text{Ind}(D) = \text{Ind}(\widehat{D}_m).$$

By the APS formula, we have

$$\text{Ind}(\widehat{D}_m) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

# The proof of the main theorem

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# Outline of the proof

## Theorem (FFMOYY arXiv:1910.01987)

For  $m \gg 0$ , we have a formula

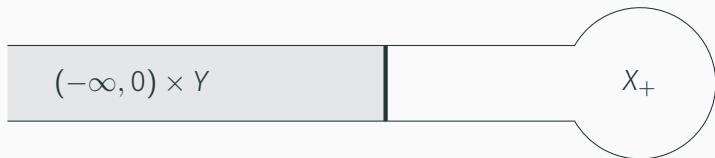
$$\text{Ind}_{\text{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

The proof is modelled on the original embedding proof of the Atiyah-Singer index theorem.

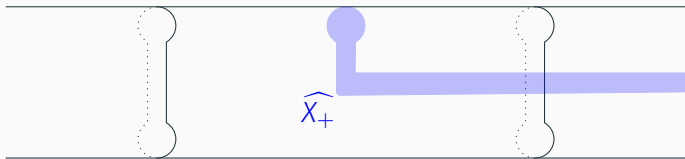
1. Embed  $\widehat{X}_+$  into  $\mathbb{R} \times X$ .
2. Extend both  $\widehat{D}$  on  $\widehat{X}_+$  and  $D + m\kappa\gamma$  on  $\{10\} \times X$  to  $\mathbb{R} \times X$ .
3. Use the product formula, the APS formula, and a Witten localisation argument.

# Embedding of $\widehat{X}_+$ into $\mathbb{R} \times X$

$$\widehat{X}_+ := (-\infty, 0] \times Y \cup X_+.$$

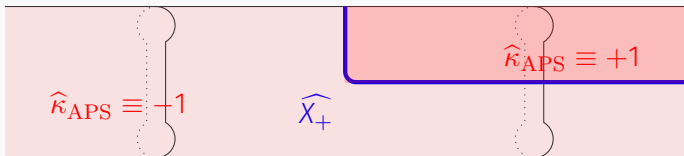


We can embed  $\widehat{X}_+$  into  $\mathbb{R} \times X$  as follows:



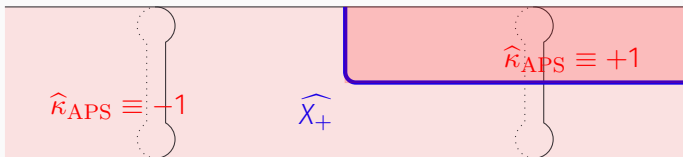
## Extension of $\widehat{D}$ and $D + m\kappa\gamma$ to $\mathbb{R} \times X$

$(\mathbb{R} \times X) \setminus \widehat{X}_+$  has two connected components. We denote by  $(\mathbb{R} \times X)_-$  the one containing  $\{-10\} \times X_+$  and by  $(\mathbb{R} \times X)_+$  the other half. Let  $\widehat{\kappa}_{\text{APS}}: \mathbb{R} \times X \rightarrow [-1, 1]$  be a step function such that  $\widehat{\kappa}_{\text{APS}} \equiv \pm 1$  on  $(\mathbb{R} \times X)_\pm$ .



We consider

$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\text{APS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\text{APS}}\gamma) - \partial_t & 0 \end{pmatrix}.$$

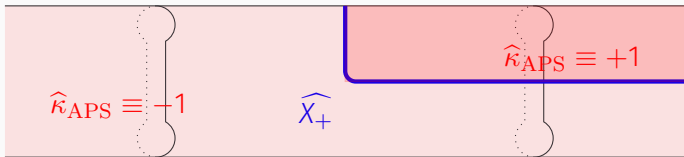


$$\hat{\mathcal{D}}_m := \begin{pmatrix} 0 & (D + m\hat{\kappa}_{\text{APS}}\gamma) + \partial_t \\ (D + m\hat{\kappa}_{\text{APS}}\gamma) - \partial_t & 0 \end{pmatrix}.$$

$$\hat{\kappa}_{\text{APS}} \equiv \kappa \text{ on } \{10\} \times X.$$

Proposition (APS formula)

$$\text{Ind}(\hat{\mathcal{D}}_m) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}$$



$$\hat{\mathcal{D}}_m := \begin{pmatrix} 0 & (D + m\hat{\kappa}_{\text{APS}}\gamma) + \partial_t \\ (D + m\hat{\kappa}_{\text{APS}}\gamma) - \partial_t & 0 \end{pmatrix}.$$

The restriction of  $\hat{\mathcal{D}}_m$  to a tubular neighbourhood of  $\hat{X}_+$  is isomorphic to

$$\begin{pmatrix} 0 & (\hat{D} + m \operatorname{sgn} \gamma) + \partial_t \\ (\hat{D} + m \operatorname{sgn} \gamma) - \partial_t & 0 \end{pmatrix}$$

on  $\mathbb{R} \times \hat{X}_+$  near  $\{0\} \times \hat{X}_+$ , where  $\hat{D}$  is the extension of  $D|_{X_+}$  to  $\hat{X}_+$ .



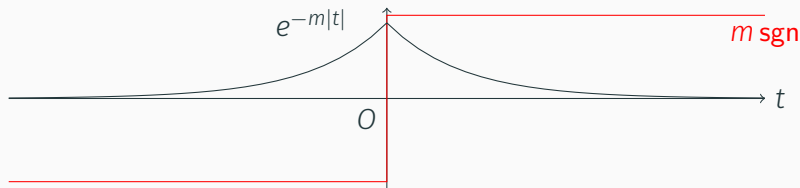
# Witten localisation

## Theorem (Witten localisation)

For  $m \gg 0$ , we have

$$\text{Ind}(\widehat{\mathcal{D}}_m) = \text{Ind} \begin{pmatrix} 0 & (\widehat{D} + m \text{sgn } \gamma) + \partial_t \\ (\widehat{D} + m \text{sgn } \gamma) - \partial_t & 0 \end{pmatrix}.$$

The proof is too technical to state here, but the idea is simple.



$$\begin{pmatrix} 0 & \partial_t + m \text{sgn} \\ -\partial_t + m \text{sgn} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e^{-m|t|} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

### Proposition (Product formula)

$$\text{Ind} \begin{pmatrix} 0 & (\widehat{D} + m \operatorname{sgn} \gamma) + \partial_t \\ (\widehat{D} + m \operatorname{sgn} \gamma) - \partial_t & 0 \end{pmatrix} = \text{Ind}(\widehat{D})$$

## Theorem (FFMOYY arXiv:1910.01987)

For  $m \gg 0$ , we have a formula

$$\text{Ind}_{\text{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

By definition, we have  $\text{Ind}_{\text{APS}}(D|_{X_+}) = \text{Ind}(\widehat{D})$ .

By the product formula, we have

$$\text{Ind}(\widehat{D}) = \text{Ind} \begin{pmatrix} 0 & (\widehat{D} + m \text{sgn } \gamma) + \partial_t \\ (\widehat{D} + m \text{sgn } \gamma) - \partial_t & 0 \end{pmatrix}.$$

By the Witten localisation argument, for  $m \gg 0$ , we have

$$\text{Ind} \begin{pmatrix} 0 & (\widehat{D} + m \text{sgn } \gamma) + \partial_t \\ (\widehat{D} + m \text{sgn } \gamma) - \partial_t & 0 \end{pmatrix} = \text{Ind}(\widehat{\mathcal{D}}_m).$$

By the APS formula, we have

$$\text{Ind}(\widehat{\mathcal{D}}_m) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$